f-Divergences and Applications to Privacy

Mario Diaz (IIMAS – UNAM)

Based on joint work with Shahab Asoodeh (McMaster) & Flavio Calmon (Harvard)

Slides available at *https://www.mariodiaztorres.com/*

Outline

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f-Divergences and Contraction

Probability Simplex

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From a statistical perspective, the geometry of $P([n])$ is not obvious...

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Some f-divergences have useful operational interpretations!

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Data Processing Inequality (DPI): $D_f(KPkKQ) \quad D_f(PkQ)$

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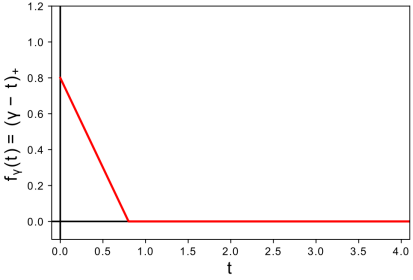

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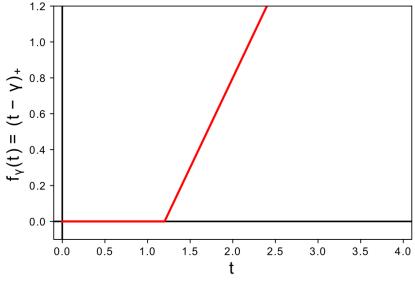

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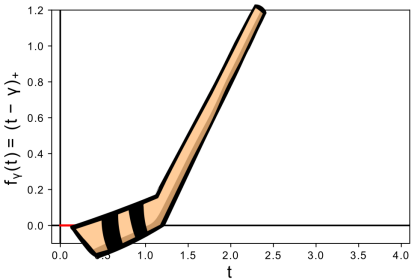

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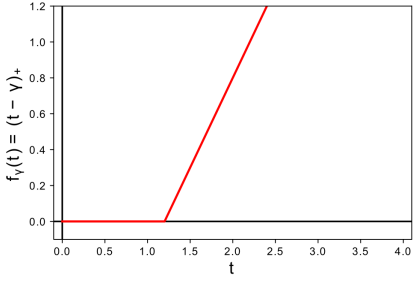

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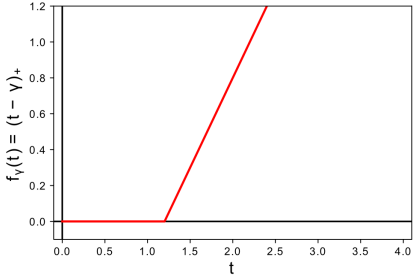

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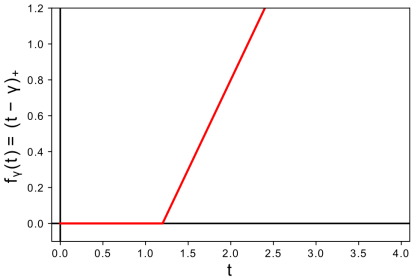

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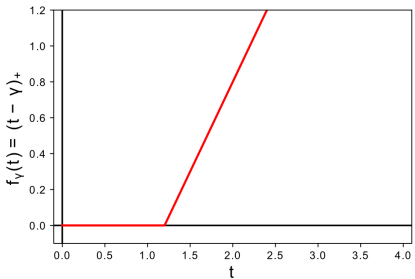

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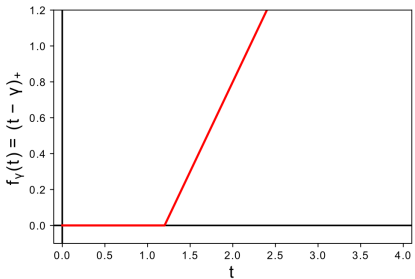

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If f is twice differentiable, then

$$D_f(P\|Q) = \int_0^{\ 1} E\ (P\|Q) f''(\ )\mathrm{d}\ .$$

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There are many relations between f-divergences...

...but there is a systematic form to obtain them!

f-Divergences V: Joint Range Strategy

P. Harremoës & I. Vajda. "On pairs of f-divergences and their joint range," *IEEE Transactions on Information Theory*, 2011.

f-Divergences V: Joint Range Strategy

Notation. Given two f-divergences D_f and D_g, their joint range is defined as

$$R(f;g) := \{(D_f(P\|Q); D_g(P\|Q)) : P;Q \text{ pmfs on some alphabet } X\};$$

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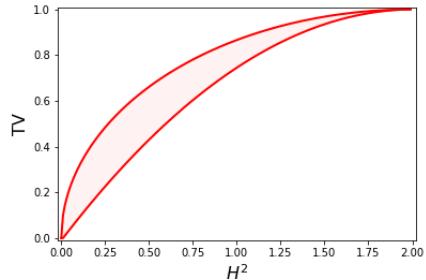

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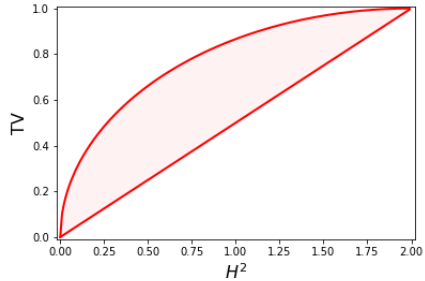

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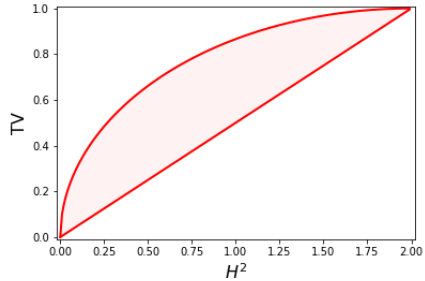

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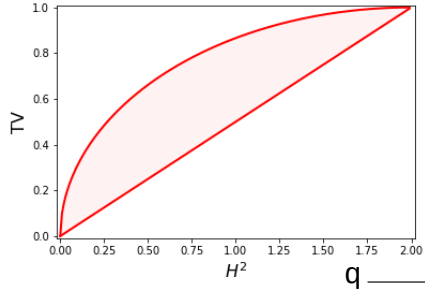

$\frac{1}{2}H^2(P;Q)$ $\quad$ $\mathrm{TV}(P;Q)$ $\quad$ $H(P;Q)$ $\quad$ 1 $\quad$ $H^2(P;Q)=4$

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f-Divergences VI: Strong Data Processing Inequality

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For all $P;Q$, we have $D_f(KPkKQ)$ $_f(K)D_f(PkQ)$.

$$P \;!\; \boxed{K} \;!\; KP$$

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$$_f(K) = \sup_{\substack{P;Q: \\ 0<D_f(PkQ)<1}} \frac{D_f(KPkKQ)}{D_f(PkQ)}:$$

For all $P;Q$, we have $D_f(KPkKQ)$ $_f(K)D_f(PkQ)$.

$$P \;!\; \boxed{K} \;!\; KP$$

$$Q \;!\; \boxed{K} \;!\; KQ$$

Thm. If K is a Markov kernel, then $_{TV}(K) = \sup_{x_1;x_2 2X} TV(K(\,jx_1)kK(\,jx_2))$.

R. Ahlswede & P. Gács. "Spreading of sets in product spaces and hypercontraction of the Markov operator," *The Annals of Probability*, 1976.
R.L. Dobrushin. "Central limit theorem for nonstationary Markov chains. I," *Theory of Probability & Its Applications*, 1956.

f-Divergences VII: Generalized Dobrushin's Formula

S. Asoodeh, M. Diaz & F. Calmon. "Contraction of E -divergence and its applications to privacy," *arXiv:2012.11035*.
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f-Divergences VII: Generalized Dobrushin's Formula

Notation. For ease of notation, we let $\coloneqq E$.

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f-Divergences VII: Generalized Dobrushin's Formula

Notation. For ease of notation, we let $\coloneqq {}_{E}$.

Thm. Let $2\ (0; 1]$. If K is a Markov kernel, then

$$(K) = \sup_{x_1;x_2 2X} E\ (K(\,jx_1)kK(\,jx_2)):$$

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Inequalities between contraction coefficients

$$(K) \quad _{\mathrm{TV}}(K) \quad 1 \quad \frac{1 \quad (K)}{} \text{ for all } \quad 1$$

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Inequalities between contraction coefficients

$$(K) \quad _{TV}(K) \quad 1 \quad \frac{1 \quad (K)}{} \text{ for all } \quad 1$$

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Local Di erential Privacy

Big Data, Big Scandals

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Ignoring privacy could be problematic...

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Ignoring privacy could be problematic...

Different settings, different precise meanings...

Privacy in Surveys

S.L. Warner. "Randomized response: A survey technique for eliminating evasive answer bias," *Journal of the American Statistical Association*, 1965.

Privacy in Surveys

Personal Information

Collected Information

Respondent

Curator

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Randomized Response Mechanism

1. Toss a coin.
2. If heads, then answer honestly.
3. If tails, then toss and answer "Yes" if heads, "No" if tails.

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Randomized Response Mechanism

1. Toss a coin.
2. If heads, then answer honestly.
3. If tails, then toss and answer "Yes" if heads, "No" if tails.

Def. For $\quad 0$, the randomized response mechanism is defined by

$K(\,j0) = \mathrm{Ber}(1 \quad p\,) \;\&\; K(\,j1) = \mathrm{Ber}(p\,);$

where $p \;= e\; =(1 + e\;)$.

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Local Di erential Privacy

A. Ev mievski, J. Gehrke & R. Srikant. \Limiting privacy breaches in privacy preserving data mining," ACM PODS, 2003.

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Notation. Recall that $K(A|x) = P(Y \in A | X = x)$ where $X \to \boxed{K} \to Y$.

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Notation. Recall that $K(A|x) = P(Y \ 2\ A|X = x)$ where X ! K ! Y.

Def. Let $\quad 0$ and $\ 2\ [0;1]$. A kernel K is $(\ ;\ )$-LDP if, for every $x_1; x_2\ 2\ X$ and $A\ \ Y$,

$$K(A|x_1) \quad e\ K(A|x_2) + \ :$$

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Example. The randomized response mechanism is $(0;\ )$-LDP.

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Three Archetypal Problems in Privacy Analysis

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LDP as Contraction of HS-Divergence

S. Asoodeh, M. Diaz & F. Calmon. \Contraction of E -divergence and its applications to privacy," arXiv:2012.11035.

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Thm. A privacy mechanism K is $(\ ;\ )$-LDP if and only if $_{e}(K)$.

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Thm. A privacy mechanism K is $(\ ;\ )$-LDP if and only if $_{e}(K)$.

Proof. K is $(\ ;\ )$-LDP , $\sup_{x_1;x_2 2X} \sup_{A\ Y} [K(Ajx_1) \quad e\ K(Ajx_2)]$

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Proof. K is $(\ ;\ )$-LDP ,

$$\sup_{x_1;x_2 \in X} \sup_{A \ Y} [K(A|x_1) \quad e\, K(A|x_2)]$$

,

$$\sup_{x_1;x_2 \in X} E_e\,(K(\ |x_1)\|K(\ |x_2))$$

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Binary Hypothesis Testing

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Goal. From a sample of X, choose between

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True Negative Rate

$$(\ ) := P(\ (X) = 0 \mid H_0)$$

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Cost of LDP in Binary Hypothesis Testing

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Notation. Let $P^{\otimes n}$ be the distribution of

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Stein's Lemma. If $\varepsilon \in (0, 1)$, then

$$\lim_{n \to \infty} \frac{1}{n} \log \frac{1}{\beta_\varepsilon(P^{\otimes n}, Q^{\otimes n})} = D_{\mathrm{KL}}(P \| Q).$$

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$$D_{KL}(KP \| KQ) \quad \varepsilon_{KL}(K) D_{KL}(P \| Q).$$

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$$D_{KL}(KP \| KQ) \leq \eta_{KL}(K) D_{KL}(P \| Q).$$

By the relation between f and β,

$$\eta_{KL}(K) \leq 1 - \frac{1 - \eta_{e^\varepsilon}(K)}{e^\varepsilon}.$$

Y. Polyanskiy & Y. Wu. Lecture Notes on Information Theory, 2014.
S. Asoodeh, M. Diaz & F. Calmon. "Contraction of E$_\gamma$-divergence and its applications to privacy," arXiv:2012.11035.

Cost of LDP in Binary Hypothesis Testing

Notation. Let P^n be the distribution of

$$X_1, \ldots, X_n \overset{\text{iid}}{\sim} P.$$

Stein's Lemma. If $\alpha \in (0,1)$, then

$$\lim_{n\to\infty} \frac{1}{n} \log \frac{1}{\beta(P^n, Q^n)} = D_{KL}(P\|Q).$$

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$$\eta_{KL}(K) \le 1 - \frac{1 - \beta_e(K)}{e^\varepsilon}.$$

Since K is (ε, δ)-LDP, $\beta_e(K) \ge \delta$,

$$\eta_{KL}(K) \le 1 - \frac{1-\delta}{e^\varepsilon}.$$

Y. Polyanskiy & Y. Wu. Lecture Notes on Information Theory, 2014.
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Cost of LDP in Binary Hypothesis Testing

Notation. Let $P^{\otimes n}$ be the distribution of

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Stein's Lemma. If $\alpha \in (0,1)$, then

$$\lim_{n\to\infty} \frac{1}{n} \log \frac{1}{\beta(P^{\otimes n}, Q^{\otimes n})} = D_{KL}(P\|Q).$$

Under (ε, δ)-LDP: $P \to KP$, $Q \to KQ$ &

$D_{KL}(KP\|KQ) \quad {}_{KL}(K) D_{KL}(P\|Q)$.

By the relation between $_f$ and ,

$$_{KL}(K) \quad 1 \quad \frac{1 \quad {}_e(K)}{e}.$$

Since K is (ε, δ)-LDP , ${}_e(K) \quad$,

$$_{KL}(K) \quad 1 \quad \frac{1}{e}.$$

Thm. If K is (ε, δ)-LDP, then $\lim_{n\to\infty} \frac{1}{n} \log \frac{1}{\beta(KP^{\otimes n}, KQ^{\otimes n})} \quad 1 \quad \frac{1}{e} \quad D_{KL}(P\|Q)$.

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Notation. Let $P^{\otimes n}$ be the distribution of

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Under (ε, δ)-LDP: $P \to KP$, $Q \to KQ$ &

$D_{\mathrm{KL}}(KP\|KQ) \quad \eta_{\mathrm{KL}}(K) D_{\mathrm{KL}}(P\|Q)$.

By the relation between f and ,

$$\eta_{\mathrm{KL}}(K) \quad 1 \quad \frac{1 \quad e^{\varepsilon}(K)}{e^{\varepsilon}}.$$

Since K is (ε, δ)-LDP , $e^{\varepsilon}(K)$,

$$\eta_{\mathrm{KL}}(K) \quad 1 \quad \frac{1}{e^{\varepsilon}}.$$

Thm. If K is (ε, δ)-LDP, then $\lim_{n\to\infty} \frac{1}{n} \log \frac{1}{\beta(KP^{\otimes n}, KQ^{\otimes n})} \quad 1 \quad \frac{1}{e^{\varepsilon}} \quad D_{\mathrm{KL}}(P\|Q)$.

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Three Archetypal Problems in Privacy Analysis

Understand privacy from a `fundamental' perspective

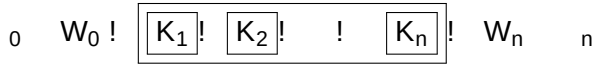
Assess the cost of privacy in speci c statistical applications

Compute/estimate the privacy parameters of speci c mechanisms

DP under Composition: Iterative Algorithms

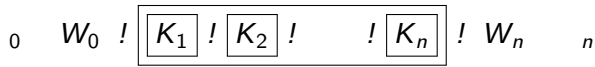
S. Asoodeh, M. Diaz & F. Calmon. \Contraction of E -divergence and its applications to privacy," arXiv:2012.11035.

DP under Composition: Iterative Algorithms



S. Asoodeh, M. Diaz & F. Calmon. \Contraction of E -divergence and its applications to privacy," arXiv:2012.11035.

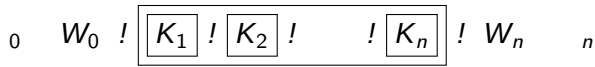
DP under Composition: Iterative Algorithms



DP for iterative algorithms: $\sup_{0;\ {}^{\theta}_{0} 2 P(X)} \mathrm{E}\ (\ {}_{n}k\ {}^{0}_{n})$ with $= e$

S. Asoodeh, M. Diaz & F. Calmon. "Contraction of E -divergence and its applications to privacy," *arXiv:2012.11035*.

DP under Composition: Iterative Algorithms



DP for iterative algorithms: $\sup_{0;\ {}^{\theta}_{0} 2P(X)} \mathrm{E}\ (\ {}_{n}k\ {}^{0}_{n})$ with $= e$

Contraction philosophy: $\mathrm{E}\ (\ {}_{n}k\ {}^{0}_{n})\quad \mathrm{E}\ (K_n)\quad \mathrm{E}\ (K_1)$

S. Asoodeh, M. Diaz & F. Calmon. "Contraction of E -divergence and its applications to privacy," *arXiv:2012.11035*.

Final Remarks

Slides available at *https://www.mariodiaztorres.com/*

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Archetypal Problems in Privacy Analysis

Understand privacy from a 'fundamental' perspective

Quantify the cost of privacy in statistical applications

Estimate the privacy parameters of specific mechanisms

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Final Remarks

Archetypal Problems in Privacy Analysis

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Quantify the cost of privacy in statistical applications

Estimate the privacy parameters of specific mechanisms

Provide statistical evidence of incorrect implementation

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Archetypal Problems in Privacy Analysis

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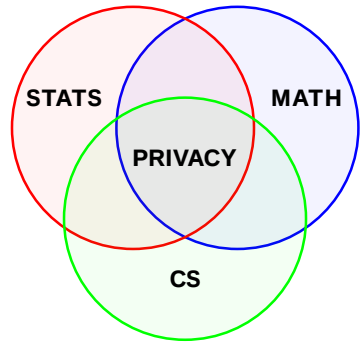

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