Operator-Valued Free Probability Theory

Part II: General Theory

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Mingo, J. A., & Speicher, R. *Free probability and random matrices*. Springer, 2017.

Last time...

- Block Gaussian matrices $\begin{pmatrix} X_N^{(1,1)} & \cdots & X_N^{(1,d)} \\ \vdots & \ddots & \vdots \\ X_N^{(d,1)} & \cdots & X_N^{(d,d)} \end{pmatrix} \to \begin{pmatrix} s_{1,1} & \cdots & s_{1,d} \\ \vdots & \ddots & \vdots \\ s_{d,1} & \cdots & s_{d,d} \end{pmatrix}$

- Semi-circular element (with covariance σ): $X := (s_{i,j})_{i,j=1}^d \in \mathrm{M}_d(\mathcal{A})$

- Moment-cumulant formula: $(\mathrm{id} \otimes \varphi)(X^m) = \sum_{\pi \in \mathrm{NC}_2(m)} \kappa_\pi$

- Cumulants: $\kappa_{(())()} = \eta(\eta(\mathrm{I}))\eta(\mathrm{I})$ with $\eta(W)_{i,j} = \sum_{k,l} W_{k,l}\sigma(i,k;l,j)$

- Cauchy transform: $G(z\mathrm{I}) = \sum_{m \geq 0} z^{-(m+1)}(\mathrm{id} \otimes \varphi)(X^m)$

- Functional equation: $zG(z\mathrm{I}) = \mathrm{I} + \eta(G(z\mathrm{I}))G(z\mathrm{I})$

Operator-valued Probability Space

"Use $\mathrm{id} \otimes \varphi : \mathrm{M}_d(\mathcal{A}) \to \mathrm{M}_d(\mathbb{C})$ instead of $\mathrm{tr}_d \otimes \varphi : \mathrm{M}_d(\mathcal{A}) \to \mathbb{C}$"

Definition. Let $\mathcal{A}$ be a unital algebra and $\mathcal{B} \subset \mathcal{A}$ a unital subalgebra. A linear map $E : \mathcal{A} \to \mathcal{B}$ is a conditional expectation if

i) $E(b) = b$ for all $b \in \mathcal{B}$;

ii) $E(b_1 a b_2) = b_1 E(a) b_2$ for all $a \in \mathcal{A}$ and $b_1, b_2 \in \mathcal{B}$.

An operator-valued probability space $(\mathcal{A}, E, \mathcal{B})$ consists of $\mathcal{B} \subset \mathcal{A}$ and a conditional expectation $E : \mathcal{A} \to \mathcal{B}$.

Example. $\mathcal{A} = \mathrm{M}_d(\mathcal{C})$, $\mathcal{B} = \mathrm{M}_d(\mathbb{C})$, and $E : \mathcal{A} \to \mathcal{B}$ given by

$$E((x_{i,j})_{i,j=1}^d) = (\varphi(x_{i,j}))_{i,j=1}^d.$$

Operator-valued Moments and Cumulants

Definition. The $\mathcal{B}$-valued moments of $a \in \mathcal{A}$ are the collection

$$\{E(b_0 a b_1 \cdots b_{n-1} a b_n) : n \in \mathbb{N}, b_0, \ldots, b_n \in \mathcal{B}\}.$$

Definition. The $\mathcal{B}$-valued free cumulants $\kappa_n^{\mathcal{B}} : \mathcal{A}^n \to \mathcal{B}$ are defined by

$$E(a_1 \cdots a_n) = \sum_{\pi \in \mathrm{NC}(n)} \kappa_\pi^{\mathcal{B}}(a_1, \ldots, a_n).$$

Note: $\kappa_\pi^{\mathcal{B}}$ 'respects' the nested (non-commutative) structure of π.

Ex. Let $\pi = \{(1,10),(2,5,9),(3,4),(6),(7,8)\}$. Then,

$$\kappa_\pi^{\mathcal{B}}(a_1, \ldots, a_{10}) = \kappa_2^{\mathcal{B}}(a_1 \kappa_3^{\mathcal{B}}(a_2 \kappa_2^{\mathcal{B}}(a_3, a_4), a_5 \kappa_1^{\mathcal{B}}(a_6) \kappa_2^{\mathcal{B}}(a_7, a_8), a_9), a_{10}).$$

Theorem. The $\mathcal{B}$-valued free cumulants are well defined.

Operator-valued Cauchy Transform

Definition. For $a \in \mathcal{A}$, we define its $\mathcal{B}$-valued Cauchy transform by

$$G_a(b) = \sum_{n \geq 0} E(b^{-1}(ab^{-1})^n),$$

and its $\mathcal{B}$-valued R transform by

$$R_a(b) = \sum_{n \geq 0} \kappa_{n+1}^{\mathcal{B}}(ab, \ldots, ab, a).$$

Comment. There is an analytic counterpart for the above definition.

Theorem. The $\mathcal{B}$-valued Cauchy and R transforms satisfy

$$bG(b) = 1 + R(G(b))G(b).$$

Operator-valued Semi-circular Elements

Definition. We say that $s \in \mathcal{A}$ is a $\mathcal{B}$-valued semi-circular element if $\kappa_n^{\mathcal{B}}(sb_1, \ldots, sb_n) = 0$ for all $n \neq 2$ and all $b_1, \ldots, b_n \in \mathcal{B}$.

Proposition. If s is a $\mathcal{B}$-valued semi-circular element, then $R(b) = \eta(b)$ where $\eta : \mathcal{B} \to \mathcal{B}$ is given by $\eta(b) = E(sbs)$. In particular,

$$bG(b) = 1 + \eta(G(b))G(b).$$

Corollary. If s is a $\mathcal{B}$-valued semi-circular element, then

$$E(s^m) = \sum_{\pi \in \mathrm{NC}_2(m)} \kappa_\pi^{\mathcal{B}}(s, \ldots, s).$$

Freeness with Amalgamation I

Definition. Let $(\mathcal{A}_i)_{i \in I}$ be a family of subalgebras of $\mathcal{A}$ containing $\mathcal{B}$. We say that $(\mathcal{A}_i)_{i \in I}$ are free with amalgamation over $\mathcal{B}$ if

$$E(a_1 \cdots a_n) = 0,$$

whenever $a_i \in \mathcal{A}_{j_i}$, $j_1 \neq \cdots \neq j_n$, and $E(a_i) = 0$ for all $i \in [n]$.

Ex. If $\{x_1, x_2\}$ and $\{y_1, y_2\}$ are free with amalgamation over $\mathcal{B}$, then

$$E(y_1 x_1 y_2) = E(y_1 E(x_1) y_2),$$
$$E(x_1 y_1 x_2 y_2) = E(x_1 E(y_1) x_2) E(y_2) + E(x_1) E(y_1 E(x_2) y_2) - E(x_1) E(y_1) E(x_2) E(y_2).$$

Freeness with Amalgamation II

Theorem. The following are equivalent:

a) x and y are free over $\mathcal{B}$;
b) mixed cumulants in x and y vanish.

Corollary. If x and y are free over $\mathcal{B}$, then $R_{x+y}(b) = R_x(b) + R_y(b)$.

Theorem. If x and y are free over $\mathcal{B}$, then

$$G_{x+y}(b) = G_x(b - R_y(G_{x+y}(b))).$$

Closing the Circle: Block Random Matrices

Proposition. Let $(\mathcal{C}, \varphi)$ be a non-commutative probability space and fix $d \in \mathbb{N}$. Define $\mathcal{A} = \mathrm{M}_d(\mathcal{C})$, $\mathcal{B} = \mathrm{M}_d(\mathbb{C})$, and $E = \mathrm{id} \otimes \varphi$. Then, the $\mathcal{B}$-valued cumulants of $A_1, \ldots, A_n \in \mathcal{A}$ are given by

$$\kappa_n^{\mathcal{B}}(A_1, \ldots, A_n)(i, j) = \sum_{i_2, \ldots, i_n = 1}^{d} \kappa_n(A_1(i, i_2), \ldots, A_n(i_n, j)).$$

Corollary. In the notation of the previous proposition, if the entries of two matrices are free in $(\mathcal{C}, \varphi)$, then the two matrices are free over $\mathcal{B}$.

Summary

- Operator-valued probability space: conditional expectation $E : \mathcal{A} \to \mathcal{B}$
- $\mathcal{B}$-valued moments, cumulants, Cauchy transform, and R transform
- Functional equation: $bG(b) = 1 + R(G(b))G(b)$
- Freeness with amalgamation over $\mathcal{B}$: $R_{x+y}(b) = R_x(b) + R_y(b)$
- Asymptotic freeness over M_d for block random matrices

Thank you!