Operator-Valued Free Probability Theory

Part I: Motivation & Semi-circular Elements

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Mingo, J. A., & Speicher, R. *Free probability and random matrices*. Springer, 2017.

Gaussian Unitary Ensemble (GUE)

Let $\{x_{i,j}, y_{i,j} : i \geq j\}$ be a family of i.i.d. standard Gaussian variables. For each $N \in \mathbb{N}$, let

$$X_N := \begin{pmatrix} x_{1,1} & \frac{x_{1,2}+\mathrm{i}y_{1,2}}{\sqrt{2}} & \cdots & \frac{x_{1,N}+\mathrm{i}y_{1,N}}{\sqrt{2}} \\ \frac{x_{1,2}-\mathrm{i}y_{1,2}}{\sqrt{2}} & x_{2,2} & \cdots & \frac{x_{2,N}+\mathrm{i}y_{2,N}}{\sqrt{2}} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{x_{1,N}-\mathrm{i}y_{1,N}}{\sqrt{2}} & \frac{x_{2,N}-\mathrm{i}y_{2,N}}{\sqrt{2}} & \cdots & x_{N,N} \end{pmatrix}.$$

Comments.

a) $X_N = X_N^*$ and $X_N \stackrel{\mathrm{d}}{=} U_N X_N$ for all $N \times N$ unitary matrix U_N

b) Asymptotic spectral distribution: semi-circle law

c) Free probability: semi-circular elements

Block Gaussian Matrices (Example)

Let A_N, B_N and C_N independent GUEs and let

$$X_N := \begin{pmatrix} A_N & B_N & C_N \\ B_N & A_N & B_N \\ C_N & B_N & A_N \end{pmatrix}.$$

Comments.

a) $X_N = X_N^*$ and $X_N \overset{\mathrm{d}}{=} (\mathrm{I}_3 \otimes U_N) X_N$ for all $N \times N$ unitary matrix U_N

b) Asymptotic spectral distribution: not trivial
(Polynomials in Gaussian matrices $\Rightarrow$ block Gaussian matrices)

c) Free probability:

$$X_N \to \left(\begin{smallmatrix} a & b & c \\ b & a & b \\ c & b & a \end{smallmatrix}\right) \text{ where } a, b, c \text{ are free semicircular variables}$$

Semi-circular Elements

Definition. Let $X = X^* \in \mathrm{M}_d(\mathcal{A})$ with

$$X = \begin{pmatrix} s_{1,1} & s_{1,2} & \cdots & s_{1,d} \\ s_{2,1} & s_{2,2} & \cdots & s_{2,d} \\ \vdots & \vdots & \ddots & \vdots \\ s_{d,1} & s_{d,2} & \cdots & s_{d,d} \end{pmatrix}.$$

We say that X is a semi-circular element with covariance σ if $(s_{i,j} : i, j \in [d])$ is a semicircular family with covariance σ, i.e.,

$$\varphi(s_{i,j} s_{k,l}) = \sigma(i, j; k, l).$$

Proposition. If $(s_{i,j} : i, j \in [d])$ is a semicircular family with covariance σ, then, for all $m \in \mathbb{N}$ and $i_1, \ldots, i_m, j_1, \ldots, j_m \in [d]$,

$$\varphi(s_{i_1,j_1} \cdots s_{i_m,j_m}) = \sum_{\pi \in \mathrm{NC}_2(m)} \prod_{u \underset{\pi}{\sim} v} \sigma(i_u, j_u; i_v, j_v).$$

Matricial Moments

Definition. The moments of X are the collection

$$\{(\mathrm{tr}_d \otimes \varphi)(X^m) : m \in \mathbb{N}\},$$

where $\mathrm{M}_d(\mathcal{A})$ is understood as $\mathrm{M}_d \otimes \mathcal{A}$.

Problem. The moments of X do not admit a recursive structure.

Definition. The matricial moments of X are the collection

$$\{(\mathrm{id} \otimes \varphi)(a_0 X a_1 \cdots a_{m-1} X a_m) : m \in \mathbb{N}, a_0, \ldots, a_m \in \mathrm{M}_d\},$$

where id is the identity function on M_d.

Matricial Cumulants

Definition. Given a semi-circular element X with covariance σ, its matricial cumulant $\kappa_\pi \in \mathrm{M}_d$ with $\pi \in \mathrm{NC}_2(m)$ is defined by

$$\kappa_\pi(i,j) := \sum_{i_1,\dots,i_{m+1}} \delta_{i,i_1}\delta_{j,i_{m+1}} \prod_{u \underset{\pi}{\sim} v} \sigma(i_u, i_{u+1}; i_v, i_{v+1}).$$

Furthermore, its covariance mapping $\eta : \mathrm{M}_d \to \mathrm{M}_d$ is defined by

$$\eta(W)(i,j) = \sum_{k,l} W(k,l)\sigma(i,k;l,j).$$

Ex. $\kappa_{(())()} = \eta(\eta(\mathrm{I}))\eta(\mathrm{I})$, $\kappa_{((()))} = \eta(\eta(\eta(\mathrm{I})))$ and $\kappa_{()()()} = \eta(\mathrm{I})\eta(\mathrm{I})\eta(\mathrm{I})$.

Moment-Cumulant Formula. If X is a semi-circular element with covariance σ, then $(\mathrm{id} \otimes \varphi)(X^m) = \sum_{\pi \in \mathrm{NC}_2(m)} \kappa_\pi$.

Matricial Cauchy Transform

Definition. Let X be a semi-circular element with covariance σ. For $z \in \mathbb{H}^+$, the matricial Cauchy transform of X is defined by

$$G(z\mathrm{I}) = \sum_{m \geq 0} z^{-(m+1)} (\mathrm{id} \otimes \varphi)(X^m).$$

Comments.

a) This definition can be extended to more general matrices.

b) If $g(z) := (\mathrm{tr}_d \otimes \varphi)((z - X)^{-1})$, then $g(z) = \mathrm{tr}_d(G(z\mathrm{I}))$.

Lemma. If X is a semi-circular element with covariance σ and $\pi = (\pi_1)\pi_2$, then

$$\kappa_\pi = \eta(\kappa_{\pi_1})\kappa_{\pi_2}.$$

Functional Equation

Theorem. For every $z \in \mathbb{H}^+$,

$$zG(z\mathrm{I}) = \mathrm{I} + \eta(G(z\mathrm{I}))G(z\mathrm{I}).$$

Comments.

a) $G(z\mathrm{I})$ is determined by this functional equation and $G(z\mathrm{I}) \in \mathbb{H}_d^-$.

b) "To isolate the correct root is not obvious at all."
Helton, Rashidi Far & Speicher (2007)

c) Let $T_z(W) := -(z\mathrm{I} + W)^{-1}$. Natural guess:

$$G(z\mathrm{I}) = -\lim_{n\to\infty} T_z(i\mathrm{I}).$$

Fixed Point Equation I

Banach Fixed Point Theorem. Let (X, d) be a non-empty complete metric space. If $T : X \to X$ is a strict contraction, then T admits a unique fixed point, i.e., there exists a unique $x^* \in X$ such that $T(x^*) = x^*$. Furthermore, $\lim_{n\to\infty} T^{\circ n}(x_0) = x^*$ for all $x_0 \in X$.

Problem. T_z might not be contractive... w.r.t. the Euclidean metric.

Definition. The Poincaré distance between two points $z_1 = x_1 + \mathrm{i}y_1$ and $z_2 = x_2 + \mathrm{i}y_2$ in $\mathbb{H}^+$ is defined as

$$d(z_1, z_2) = 2\log\left[\frac{\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} + \sqrt{(x_2 - x_1)^2 + (y_2 + y_1)^2}}{2\sqrt{y_1 y_2}}\right].$$

Fixed Point Equation II

$$T_z(w) := -(z + w)^{-1}$$

Theorem. For every $z \in \mathbb{H}^+$, T_z has a unique fixed point w^*. Furthermore, $\lim_{n\to\infty} T_z^{\circ n}(w_0) = w^*$ for all $w_0 \in \mathbb{H}^+(\mathrm{M}_d)$.

Earle-Hamilton Fixed Point Theorem. Let X be a complex Banach space and $D \subset X$. If $h : D \to D$ is a bounded holomorphic function such that $h(D)$ lies strictly inside D, then h is a strict contraction w.r.t. the Carathéodory distance.

Helton, J. W., Rashidi Far, R., & Speicher, R. Operator-valued semicircular elements: solving a quadratic matrix equation with positivity constraints. *IMRN*, 2007.

Summary

- Block Gaussian matrices and semi-circular elements
- Moments, matricial moments and matricial cumulants
- Key idea: use $\mathrm{id} \otimes \varphi$ instead of $\mathrm{tr}_d \otimes \varphi$
- Functional and fixed point equations for the matricial Cauchy trans.

Thank you!