Yamada-Watanabe theorem in the fractional case

Antoine Bordas

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ENS Paris-Saclay & CIMAT

Introduction

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where

- $g, h, b : \mathbb{R} \to \mathbb{R}$
- $X_0 \in \tilde{\mathcal{S}}_p$
- X_t is an $\mathcal{S}_p$-valued process
- B^H is a matrix fractional Brownian motion with $H > \frac{1}{2}$

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Existence? Uniqueness?

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Yamada-Watanabe theorem

The original one

J. Math. Kyoto Univ. (JMKYAZ)
11-1 (1971) 155-167

On the uniqueness of solutions of stochastic differential equations

By

Toshio YAMADA and Shinzo WATANABE

(Received September 21, 1970)

Introduction

In this paper, we shall discuss the uniqueness problem for solutions of stochastic differential equations.

The theory of stochastic differential equations, as is well known, was developed mainly by Ito and furnishes a very important tool of constructing diffusion processes. Skorohod [4] showed the existence of solutions under the condition that coefficients are only continuous and then, the problem of the uniqueness of solutions becomes important. In order to define a diffusion process through a solution of the stochastic differential equation, it is sufficient to verify *the uniqueness in the sense of the probability law of solutions*. It may be needless to say that there are many means to verify it; in analytic way, through the theory of differential equations (cf. Stroock-Varadhan [5]) and in probabilistic way through several transformations such as time change or the change of drift.

Here, we shall study mainly *the pathwise uniqueness of solutions*. In Ito's classical theory where the coefficients are assumed to be Lipschitz continuous, the pathwise uniqueness holds and the solution can be constructed on a given Brownian motion through successive approximation. The uniqueness in the sense of the probability law is obvious in this case. There are several examples where Ito's theory

$$dx_t = \sigma(x_t)dB_t + b(x_t)dt$$

with B_t a Brownian motion, $|\sigma(x) - \sigma(y)|^2 \leq \rho(|x-y|)$, $\int_0^{+\infty} \rho^{-1}(x)dx = \infty$ and b Lipschtiz continuous

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Pathwise uniqueness

Multidimensional version

Theorem (Graczyk, Malecki)

Denote by B_t a $p \times p$ Brownian matrix and consider the matrix SDE on $\mathcal{S}_p$

$$dX_t = g(X_t)dB_t h(X_t) + h(X_t)dB_t^T g(X_t) + b(X_t)dt$$

where $g, h, b : \mathbb{R} \to \mathbb{R}$ and $X_0 \in \tilde{\mathcal{S}}_p$. Suppose that

$$|g(x)h(x) - g(y)h(y)|^2 \leq \rho(|x - y|)$$

where ρ is a function such that $\int_0^\infty \rho^{-1}(x)dx = \infty$, b is locally Lipschitz and $G(x, y) = g^2(x)h^2(y) + g^2(y)h^2(x)$ is locally Lipschitz and strictly positive on $\{x \neq y\}$.

Then the pathwise uniqueness holds, up to the collision time.

[1] Graczyk, P., & Małecki, J. (2013). *Multidimensional Yamada-Watanabe theorem and its applications to particle systems.* Journal of Mathematical Physics, 54(2), 021503.

Sketch of proof

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⚠ Non colliding starting point

Particle systems

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Strong solutions of non-colliding particle systems*

Piotr Graczyk[†] Jacek Małecki[‡]

Abstract

We study systems of stochastic differential equations describing positions $x_1, \dots, x_p$ of p ordered particles, with inter-particles repulsions of the form $\frac{H_{ij}(x_i, x_j)}{x_i - x_j}$. We show the existence of strong and pathwise unique non-colliding solutions of the system with a colliding initial point $x_1(0) \leq \dots \leq x_p(0)$ in the whole generality, under natural assumptions on the coefficients of the equations.

Keywords: stochastic differential equation; strong solution; non-colliding particle system.
AMS MSC 2010: 60J60; 60H15.
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Supersedes arXiv:1407.1329.

1 Introduction

Consider the following system of SDEs

$$dx_i = \sigma_i(x_i)dB_i + \left(b_i(x_i) + \sum_{j \neq i} \frac{H_{ij}(x_i, x_j)}{x_i - x_j} \right) dt, \quad i = 1, \dots, p, \qquad (1.1)$$
$$x_1(t) \leq \dots \leq x_p(t), \quad t \geq 0,$$

describing positions of p ordered particles evolving in $\mathbb{R}$. Here $(B_i)_{i=1,\dots,p}$ denotes a collection of one-dimensional independent Brownian motions. Throughout the whole paper we assume that the coefficients of the equations are continuous and that the functions H_{ij} are non-negative and symmetric in the sense (2.1).

The SDEs systems (1.1) contain the following ones

$$dx_i = 2g(x_i)h(x_i)dB_i + \beta \left(b(x_i) + \sum_{j \neq i} \frac{G(x_i, x_j)}{x_i - x_j} \right) dt, \quad i = 1, \dots, p, \qquad (1.2)$$

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[†]LAREMA, Université d'Angers, France. E-mail: piotr.graczyk@univ-angers.fr
[‡]Faculty of Fundamental Problems of Technology, Department of Mathematics, Wrocław University of Technology, Poland. E-mail: jacek.malecki@pwr.edu.pl

$$X = (x_1, \dots x_p)$$

- Symmetric polynomials of particles:
$$e_n(X) = \sum_{i_1 < \dots < i_n} x_{i_1} \dots x_{i_n}$$

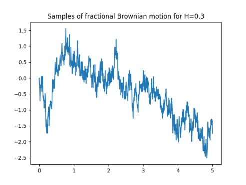
- Symmetric polynomials of square of differences:
$V_n = e_n(A)$ with $A = \{(x_i - x_j)^2 : 1 \leq i < j \leq p\}$

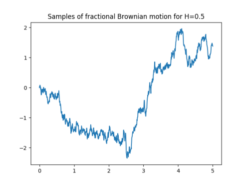
Fractional Brownian Motion

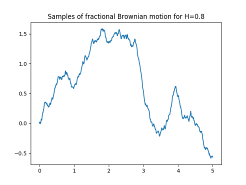
Definition

Definition

A centered Gaussian process $B = (B_t)_{t \geq 0}$ is called a ***fractional Brownian motion*** *(fBm) of Hurst parameter $H \in]0, 1[$ if it has the covariance function $R_H(t,s) = E(B_t B_s) = \frac{1}{2}\left(s^{2H} + t^{2H} - |t-s|^{2H}\right)$.*







Main properties

- Stationary increments
- Non independent increments

[2] Nualart, D. (2006). *Fractional Brownian motion: Stochastic, calculus and applications.* Proceedings oh the International Congress of Mathematicians, Vol. 3, 2006-01-01, ISBN 978-3-03719-022-7, pags. 1541-1562. 3.

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Theorem

The fractional Brownian motion is a semi-martingale if and only if $H = \frac{1}{2}$.

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Ito's calculus non usable

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Stochastic integration

In the case $H > \frac{1}{2}$:

- Pathwise approach: Young integral for processes with $\gamma-$Holder trajectories where $\gamma > 1 - H$
- Malliavin calculus: Skorohod integral

[3] Nualart, D. (2006). *The Malliavin calculus and related topics* (Vol. 1995). Berlin: Springer.

Ito's formula

Theorem

For a N-dimensional fractional Brownian motion $B^H(t)$ and for a function $F \in \mathcal{C}^2\left(\mathbb{R}^N\right)$, we have:

$$F(B^H(t)) = F(0) + \sum_{i=1}^{N} \int_0^t \frac{\partial F}{\partial x_i}(B^H(s))\delta B_i^H(s) + H \sum_{i=1}^{N} \int_0^t \frac{\partial^2 F}{\partial x_i^2}(B^H(s)) s^{2H-1} ds.$$

where $\int_0^t \frac{\partial F}{\partial x_i}(B^H(s))\delta B_i^H(s)$ stands for the Skorohod integral.

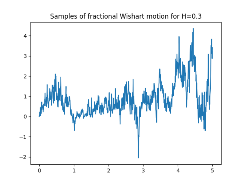
[4] Pardo, J. C., Pérez, J. L., & Pérez-Abreu, V. (2017). *On the non-commutative fractional Wishart process.* Journal of Functional Analysis, 272(1), 339-362.

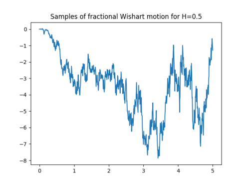
Fractional Wishart process

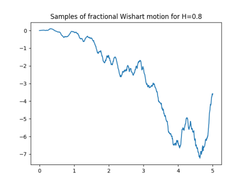
A matrix process

Definition

*Let $(B^H(t))_{t\geq 0}$ be the matrix fractional Brownian motion with parameter H. We define the **fractional Wishart process** of order n and parameter H the process $(X(t))_{t\geq 0}$ satisfying $X(t) = \left(B^H(t)\right)^T B^H(t)$ for all $t \geq 0$.*







Hadamard's formula

Theorem

Let A be a self-adjoint matrix which depends smoothly on a parameter t, that has simple spectrum. We denote by λ_j the eigenvalues and v_j the eigenvectors. Then we have the evolution equations:

$$\dot{\lambda}_k = v_k^* \dot{A} v_k$$

$$\dot{v}_k = \sum_{j \neq k} \frac{v_j^* \dot{A} v_k}{\lambda_k - \lambda_j} v_j + c_k v_k$$

$$\ddot{\lambda}_k = v_k^* \ddot{A} v_k + \sum_{j \neq k} \frac{|v_k^* \dot{A} v_j|^2}{\lambda_k - \lambda_j}$$

[4] Tao, T. (2012). *Topics in random matrix theory* (Vol. 132). American Mathematical Soc..

Spectral dynamic

Theorem

Let X be the fractional Wishart process of order n and parameter $H > \frac{1}{2}$, $\lambda_1, ..., \lambda_n$ its eigenvalues. We denote by ϕ_i the functions such that $\lambda_i(t) = \phi_i(B^H(t))$.
Then for any i and $t > 0$, we have:

$$\lambda_i(t) = \lambda_i(0) + \sum_{k=1}^{p}\sum_{h=1}^{n}\int_0^t \frac{\partial \phi_i}{\partial b_{kh}}(B^H(s))\delta b_{kh}(s)$$
$$+ 2H\int_0^t \left(p + \sum_{i \neq j} \frac{\lambda_i(s) + \lambda_j(s)}{\lambda_i(s) - \lambda_j(s)} \right) s^{2H-1} ds$$

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Matricial diffusion

Theorem

The fractional Wishart process of order n and parameter $H > \frac{1}{2}$, $(W^H(t))_{t \geq 0}$ *satisfies the stochastic differential equation:*

$$dW^H(t) = \sqrt{W^H(t)}\delta B^H(t) + \delta(B^H(t))^T\sqrt{W^H(t)} + 2Hnt^{2H-1}Idt$$

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Generalization to a non-integer order $\alpha \in \mathbb{R}$.

$$dW^H(t) = \sqrt{W^H(t)}\delta B^H(t) + \delta(B^H(t))^T\sqrt{W^H(t)} + 2H\alpha t^{2H-1}Idt$$

Outcome

Directions to go

- Fractional Wishart process with non-integer order
    - Eigenvalues dynamic
    - Uniqueness

Directions to go

- Fractional Wishart process with non-integer order

  Eigenvalues dynamic

  Uniqueness

- Yamada-Watanabe type theorem

  Eigenvalues diffusion

  Existence and uniqueness

Thank You

Bibliography

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